RESEARCH ARTICLE

Advanced Science Letters
Vol. 23, 11991–11994, 2017

Some Convergence Theorems on δ-Denjoy Integral

Manuharawati

Department of Mathematics, Universitas Negeri Surabaya, Jalan Ketintang, Surabaya 60231, Indonesia

Several descriptive integrals were contructed by using the concept of absolute continuity on an interval $[a, b] \subset R$. By generalizing interval $[a, b]$ into fundamental δ—interval for a positive function δ on interval $[a, b]$, we can construct an absolute δ—continuous function on interval $[a, b]$. In this paper, the author used literature review to construct a Denjoy integral with respect to function δ which is called δ—Denjoy integral. In addition, the author proved the Cauchy theorem and Harnach theorem on such an integral.

Keywords: Cauchy Theorem, δ—Denjoy Integral, Harnach Theorem.

1. INTRODUCTION

There have been many descriptive integrals studied by such mathematicians.[2, 3, 5, 7–9, 11, 12, 15, 16]

Using absolute continuity concept which involves interval system[1, 5, 11, 16] there have been studied Denjoy integral on a cell $[a, b] \subset R$. Schwabik[15] introduces descriptive integral using generalized ordinary differential equation approach while the others, using approximately interval system.

Quite different from the afore mentioned authors, we will study descriptive integral, especially Denjoy integral type that the absolute continuity concept used refers to absolute continuity,[13, 14] that is, the absolute continuity which involves δ—fundamental interval system.[4, 10] Beside that, we will study the Cauchy theorem and Harnach theorem on this integral later.

2. BASIC CONCEPT

In this section, we study some basic concepts and related properties that is useful for later discussion.

Definition 2.1 (Darmawijaya, 1991; Manuharawati, 2016). Let $a, b \in R, a \neq b$, δ be a positive function defined on an interval $[a, b] \subset R$, and $x \in [a, b]$. An interval $(u, v) = \{y \in [a, b]\colon x - \delta(x) \leq u < y < v \leq x + \delta(x)\}$ is called a δ—fundamental interval at x. A collection of all fundamental δ—intervals at x, denoted by D_x and has properties

A1. For every $D_x \in D_x$ and $s < x < t$,

$$D'_x = D_x \cap (s, t) \in D_x$$

A2. If $D'_x, D''_x \in D_x$, then

$$D'_x \cap D''_x \in D_x$$

A3. If A is index set and $D^\alpha_x \in D_x$ for every $\alpha \in A$, then

$$D_x = \bigcup_{\alpha \in A} D^\alpha_x \in D_x$$

A4. For every $D_x \in D_x$, D_x contains x and there exists $s, t \in D_x$ such that

$$s < x < t$$

A5. If $D_x \in D_x, u, v \in D_x$ and $u < x < v$, then there are $u_1, v_1 \in D_x$ with

$$u < u_1 < x < v_1 < v$$

Note that $D_x \neq \emptyset$. So, we can take exactly one $D_x \in D_x$. The collection of all D_x such as denoted by G_x.

Definition 2.2 (Darmawijaya, 1991). A collection Δ of all closed intervals $[u, v]$ with $u, v \in D_x \in G$ is called a Δ—fundamental full cover $(FFC - \Delta)$ on $[a, b]$ and G is called a generator of $FFC - \Delta$.

Definition 2.3 (Indrati, 1992). Let $a, b \in R, a \neq b$, δ be a positive function defined on an interval $[a, b] \subset R$, and $A \subset R$. The number $p \in R$ is said to be δ—limit point of a set A if for every fundamental δ—interval $D_p \in D_p$ we have

$$D_p \cap A - p \neq \emptyset$$

Definition 2.4 (Indrati, 1992). Let $a, b \in R, a \neq b$, δ be a positive function defined on an interval $[a, b] \subset R$, $F\colon [a, b] \to R$, and x_0 be δ—limit point of $[a, b]$. The number $l \in R$ is called to be δ—limit of a function f for x closed to x_0, denoted by

$$\delta - \lim_{x \to x_0} f(x) = l$$

if for every real number $\varepsilon > 0$, there exists a fundamental δ—interval $D_{x_0} \in D_{x_0}$ such that for every $u \in D_{x_0}, u \neq x_0$ we have

$$|F(u) - l| < \varepsilon$$

Definition 2.5 (Indrati, 1992). Let $a, b \in R, a \neq b$, δ be a positive function defined on an interval $[a, b] \subset R$, $F: [a, b] \to R$, and $x_0 \in [a, b]$ be δ—limit point of $[a, b]$. A function F is said to be δ—continuous at x_0 if

$$\delta - \lim_{x \to x_0} F(x) = F(x_0)$$

Definition 2.6 (Indrati, 1992). Let δ be a positive function defined on an interval $[a, b] \subset R$ and $x_0 \in [a, b]$. δ—Derivative of a function $F \to R$ at x_0, denoted by $D_\delta F(x_0)$ defined by

$$D_\delta F(x_0) = \delta - \lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0}$$

(if the limit exists).

Let $[a, b]$ be an interval on R with $a < b$ and $\delta: [a, b] \to R^+$. Based on the fundamental δ—interval, and the concept of an absolute continuity of a function, it was constructed some types of absolute continuity as follow.

Definition 2.7 (Manuharawati, 2013). Given a positive function $\delta: [a, b] \to R^+$, a set $X \subset R$, and a function $F: [a, b] \to R$.

(i) A function F is said to be weakly absolutely δ—continuous on X if for any real number $\varepsilon > 0$, there exists a real number $\gamma > 0$ such that for every sequence (x_i) on X there is a sequence of nonoverlapping fundamental δ—intervals (D_{x_i}), $D_{x_i} \in D_{x_i}$ such that if $u_i \in D_{x_i}$ with $\sum_i |x_i - u_i| < \gamma$ then

$$\sum_i |F(x_i) - F(u_i)| < \varepsilon$$

The set of all weakly absolutely δ—continuous functions on X is denoted by $AC_\delta(X)$.

(ii) A function F is said to be generalized weakly absolutely δ—continuous on X if there is a sequence of sets (X_i) such that $X = \cup_i X_i$ and $F \in AC_\delta(X_i)$ for every i. The set of all generalized weakly absolutely δ—continuous functions on X is denoted by $ACG_\delta(X)$.

(iii) A function F is said to be strongly absolutely δ—continuous on X if for any real number $\varepsilon > 0$, there exists a real number $\gamma > 0$ such that for every sequence (x_i) on X there is a sequence of nonoverlapping fundamental δ—intervals (D_{x_i}), $D_{x_i} \in D_{x_i}$ such that if $u_i \in D_{x_i}$ with $\sum_i |x_i - u_i| < \gamma$ then

$$\sum_i \omega(F; [a_i, b_i]) < \varepsilon$$

with $a_i = \min\{x_i, u_i\}$ and $b_i = \max\{x_i, u_i\}$. The set of all strongly absolutely δ—continuous functions on a set $X \subset [a, b]$ is denoted by $AC^*_\delta(X)$.

(iv) A function F is said to be generalized strongly absolutely δ—continuous on a set X if there exists a sequence of sets (X_i) such that $X = \cup_i X_i$ and $F \in AC^*_\delta(X_i)$ for every i. The set of all generalized strongly absolutely δ—continuous functions on X is denoted by $ACG^*_\delta(X)$.

The following theorem will be used in next discussion.

Theorem 2.1 (Manuharawati, 2013). *Let δ be a positive real function on $[a, b]$. The $C_\delta[a, b]$, $AC^*_\delta[a, b]$, and $ACG^*_\delta[a, b]$ are linear spaces.*

Theorem 2.2 (Manuharawati, 2013). *Let δ be a positive real function on $[a, b]$, $F: [a, b] \to R$ be a function and $c \in [a, b]$. If $D_\delta F(c)$ exists, then F is δ—continuous at c.*

Theorem 2.3 (Manuharawati, 2016). *If $D_\delta F(x)$ exists for every $x \in [a, b]$, then $F \in ACG^*_\delta[a, b]$.*

Theorem 2.4 (Manuharawati, 2016). *If $D_\delta F(x)$ exists nearly everywhere on $[ab]$, then $F \in ACG^*_\delta[a, b]$.*

3. RESULTS AND DISCUSSION

Definition 3.1. Let δ be a positive function defined on $[a, b] \subset R$. A function $f: [a, b] \to \underline{R}$ is said to be δ—Denjoy integrable or D^*_δ—integrable on $[a, b]$ if there exists a function $F: [a, b] \to \underline{R}$ such that $D_\delta F(x) = f(x)$ a.e. on $[a, b]$ and $F \in C_\delta[a, b] \cap ACG^*_\delta[a, b]$. The number $(D^*_\delta) \int_a^b f(t)\,dt = F(b) - F(a)$ is called D^*_δ—integral value of f on $[a, b]$ and F is called D^*_δ—primitive of f on $[a, b]$. The set of all D^*_δ—integrable functions on $[a, b]$ is denoted by $D^*_\delta[a, b]$.

It is easy to understand that the D^*_δ—integral value of f on $[a, b]$ is unique.

Theorem 3.1. *If $f, g \in D^*_\delta[a, b]$ then $f + g \in D^*_\delta[a, b]$ and $\alpha f \in D^*_\delta[a, b]$ for any real number α.*

Proof. Let F and G be D^*_δ—primitive of f and g on $[a, b]$. Then $D_\delta F(x) = f(x)$ a.e. on $[a, b]$, $D_\delta G(x) = g(x)$ a.e. on $[a, b]$, and $F, G \in C_\delta[a, b] \cap ACG^*_\delta[a, b]$. By the Definition 2.6, Theorems 2.1 and 2.2, $D^*_\delta[a, b]$ is a linear space, i.e.: $f + g \in D^*_\delta[a, b]$ and $\alpha f \in D^*_\delta[a, b]$. ■

Theorem 3.2. *If $f \in D^*_\delta[a, b]$ and $[u, v] \subset [a, b]$, then $f \in D^*_\delta[u, v]$.*

Proof. Let F be a D^*_δ—primitive of f on $[a, b]$. So, $F: [a, b] \to R$ with $D_\delta F(x) = f(x)$ a.e. on $[a, b]$ and $F \in C_\delta[a, b] \cap ACG^*_\delta[a, b]$. Since $[u, v] \subset [a, b]$, then, $F: [u, v] \to R$ with $D_\delta F(x) = f(x)$ a.e. on $[u, v]$ and $F \in C_\delta[u, v] \cap ACG^*_\delta[u, v]$. Since $F \in ACG^*_\delta[a, b]$ then there exists a sequence of sets (X_i) such that $\cup_{i \in N} X_i = [a, b]$ and $F \in AC^*_\delta(X_i)$ for every i. For every i, define

$$Y_i = X_i \cap [u, v]$$

Then we have: $[u, v] = \cup_{i \in N} Y_i$ and $F \in AC^*_\delta(Y_i)$ for every i. So, $F \in ACG^*_\delta[u, v]$. Consequently $F \in D^*_\delta[u, v]$. ■

Theorem 3.3. *If $f \in D^*_\delta[a, b]$ and $a < c < b$, then*

$$(D^*_\delta) \int_a^b f\,dt = (D^*_\delta) \int_a^c f(t)\,dt + (D^*_\delta) \int_c^b f(t)\,dt$$

Proof. By Theorem 3.1, $f \in D^*_\delta[a, c]$ and $f \in D^*_\delta[c, b]$. If F is a D^*_δ—primitive of f on $[a, b]$, then

$$\begin{aligned}(D^*_\delta) \int_a^b f(t)\,dt &= F(b) - F(a) \\ &= [F(b) - F(c)] + [F(c) - F(a)] \\ &= (D^*_\delta) \int_a^c f(t)\,dt + (D^*_\delta) \int_c^b f(t)\,dt \quad ■\end{aligned}$$

Theorem 3.4. (*Cauchy Theorem*): *If $f \in D^*_\delta[c, d]$ for every $[c, d] \subset [a, b]$ with $a < c < d < b$ and*

$$\delta - \lim_{c \to a^+; d \to b^-} (D^*_\delta) \int_a^b f(t)\,dt = A(exist)$$

then $f \in D^*_\delta[a,b]$ *with*

$$(D^*_\delta)\int_a^b f(t)\,dt = A$$

Proof. By the Theorem 3.2, we can define a function F on $[a,b]$ with $F(x) = (D^*_\delta)\int_a^x f(t)\,dt$ for every $x \in [a,b]$. So, we have $D_\delta F(x) = f(x)$ a.e. on $[a,t]$ and $F \in C_\delta[a,t] \cap ACG^*_\delta[a,t]$. $t \in [a,b]$ for every $t \in [a,b]$.

Since

$$\delta - \lim_{x\to b^-} F(x) = \delta - \lim_{;x\to b^-} (D^*_\delta)\int_a^b f(t)\,dt = A$$

then for every real number $\varepsilon > 0$ there exists a δ—fundamental interval $D_b \in D_b$ such that for every $u \in D_b$ with $u < b$ we obtain

$$|F(u) - A| < \frac{\varepsilon}{2^{k+2}}$$

for every positive integer k.

Furthermore, define $F(b) = A$. By the Definitions 2.6 and 2.7, we have: F is δ—continuous at b and

$$|F(u) - F(b)| < \frac{\varepsilon}{2^{k+2}}$$

for every positive integer k.

Hence

$$\omega(F;[u,b]) < \frac{\varepsilon}{2^{k+2}} \tag{3.1}$$

Since $t \in [a,b]$, then we choose $t = u$ such that (3.1) holds. Since $F \in ACG^*_\delta[a,u]$, then there exists a sequence of set (X_i) with $[a,u] = \cup_{i\in N} X_i$ and $F \in AC''_\delta(X_i)$ for every i. So, there exists a positive integer number m such that $u \in X_m$ and $F \in AC^*_\delta(X_m)$. Consequently, there exists a positive real number γ such that for every sequence (x_i) on X_m there exists a nonoverlapping sequence of δ—fundamental intervals (D_{x_i}) with $D_{x_i} \in D_{x_i}$ such that for every $u_i \in D_{x_i} \cap X_m$ with $\sum_{i\in N}|x_i - u_i| < \gamma$ we have

$$\sum_{i\in N}\omega(F;[a_i,b_i]) < \frac{\varepsilon}{2^{k+1}} \tag{3.2}$$

with $a_i = \min\{x_i,u_i\}$ and $b_i = \max\{x_i,u_i\}$. Furthermore, define a set $X = X_m \cup [u,b]$. If (X_i) is any sequence on X, then there exists a nonoverlapping sequence of δ—fundamental intervals (D_{x_i}) with $D_{x_i} \in D_{x_i}$, and by (3.1) and (3.2), if $u_i \in D_{x_i} \cap X$ and $\sum_{i\in N}|x_i - u_i| < \gamma$, we have

$$\begin{aligned}\sum_{i\in N}\omega(F;[a,b]) &\le \sum_{i\in N, x_i\in X_m}\omega(F;[a_i,b_i])\\ &+ \sum_{i\in N, x_i\in(u,b]}\omega(F;[a_i,b_i]) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

with $a_i = \min\{x_i,u_i\}$ and $b_i = \max\{x_i,u_i\}$. So, $F \in AC^*_\delta(X)$. Consequently, $F \in ACG^*_\delta[a,b]$.

Since the function F is δ—continuous at b, we have

$$\delta - \lim_{x\to b} F(x) = \delta - \lim_{x\to b^-} F(x) = A$$

So, we have

$$(D^*_\delta)\int_a^b f(t)\,dt = A \quad \blacksquare$$

Theorem 3.5. (*Harnach Theorem*): *Let E be a closed subset of $[a,b]$ and $([a_i,b_i])$ be a sequence of contiguous closed intervals of E on $[a,b]$. If I_E is the a characteristic function on E and $fI_E \in D^*_\delta\{a,b\}$, $f \in D^*_\delta[a_i,b_i]$ with*

$$\sum_{i\in N}(D^*_\delta)\int_{a_i}^{b_i} f(t)\,dt < \infty$$

then $f \in D^*_\delta[a,b]$ *with*

$$(D^*_\delta)\int_a^b f(t)\,dt = \sum_{i\in N}(D^*_\delta)\int_{a_i}^{b_i} f(t)\,dt + (D^*_\delta)\int_a^b fI_E(t)\,dt$$

Proof. Since $fI_E \in D^*_\delta\{a,b\}$ then there exists a function $G\colon [a,b] \to R$ such that $D_\delta G(x) = fI_E(x)$ a.e. on $[a,b]$ and $G \in C_\delta[a,b] \cap ACG^*_\delta[a,b]$. Since $f \in D^*_\delta[a_i,b_i]$, there exists a function $F_i\colon [a_i,b_i] \to R$ for every i such that $F_i \in C_\delta[a_i,b_i] \cap ACG^*_\delta[a_i,b_i]$. Since $([a_i,b_i])$ is a sequence of contiguous closed interval of E on $[a,b]$, we have

$$[a,b] - E = \bigcup_{i\in N}[a_i,b_i]$$

Furthermore, if we define a function $F\colon [a,b] \to R$ where

$$\begin{aligned}F(x) = \Bigg\{ G(x) + \sum_{i=1}^{k-1} F_i(x);\quad x = a_k;\quad a_k = b_{k-1} \quad \text{or}\\ a_k \ne b_{k-1} G(x);\quad x \in E \text{ and there is no } k \text{ with}\\ x \ge a_k G(x) + \sum_{i=1}^{k-1} F_i(x) + F_k(x);\quad x \in [a_k,b_k]\Bigg\}\end{aligned}$$

then we have:

(i) $D_\delta F(x) = f(x)$ a.e. on $[a,b]$.
(ii) $F \in AC^*_\delta(E) \cap ACG^*_\delta[a,b]$.
(iii) Since $F \in C_\delta[a_i,b_i] \cap ACG^*_\delta[a_i,b_i]$ then $F \in C_\delta(\cup_{i\in N}[a_i,b_i]) \cap ACG^*_\delta(\cup_{i\in N}[a_i,b_i])$, so $F \in C^*_\delta[a,b] \cap ACG^*_\delta[a,b]$

From (i) and (iii), we have $f \in D^*_\delta[a,b]$.

Note that

$$(D^*_\delta)\int_a^b (fI_E)(t)\,dt = (D^*_\delta)\int_E f(t)\,dt$$

Since $\sum_{i\in N}(D^*_\delta)\int_{a_i}^{b_i} f(t)\,dt < \infty$, we obtain

$$\begin{aligned}(D^*_\delta)\int_a^b f(t)\,dt &= (D^*_\delta)\int_E f(t)\,dt + (D^*_\delta)\int_{\cup_{i\in N}[a_i,b_i]} f(t)\,dt\\ &= (D^*_\delta)\int_a^b (fI_E)(t)\,dt\\ &+ \sum_{i\in N}\left[(D^*_\delta)\int_{a_i}^{b_i} f(t)\,dt\right] \quad \blacksquare\end{aligned}$$

4. CONCLUSIONS

Given a positive function δ on cell $[a,b]$ and based on the concept of δ—fundamental continuity and generalized δ—fundamental continuity, we have constructed descriptive-type integral which was defined as δ—Denjoy integral and also successfully proved the Cauchy theorem and the Harnach theorem on the integral.

Acknowledgments: I would like to thank Faculty of Mathematics and Natural Sciences for giving me the opportunity joining the manuscript clinic.

References and Notes

1. R. G. Bartle, Introduction to Real Analysis, 3rd edn., John Wiley and Sons, New York **(2000)**.
2. P. S. Bullen, *J. Aust. Math. Soc, Ser. A* 35, 236 **(1983)**.
3. J. C. Burkill, *Math. Zeit* 34, 270 **(1931)**.
4. S. Darmawijaya, Integral Dasar Deskriptif, Ph.D. Thesis, FMIPA-Universitas Gadjah Mada, Yogyakarta **(1991)**.
5. R. A. Gordon, *Amer. Math. Soc.* 4 **(1994)**.
6. R. Henstock, Linear Analysis, Butterworths, London **(1983)**.
7. D. K. Ganguly and R. Mukherjee, *Real Anal. Exch.* 31, 373 **(2005/2006)**.
8. D. K. Ganguly and R. Mukherjee, *Math. Slovaca* 58, 31 **(2008)**.
9. D. K. Ganguly and R. Mukherjee, *J. Indones. Math. Soc.* 17, 49 **(2011)**.
10. Ch. R. Indrati, Pendekatan Baru Tentang Kalkulus, Laporan Penelitian, FMIPA Universitas Gadjah Mada, Yogyakarta **(1992)**.
11. Y. Kubota, *Math. Japonica* 24, 507 **(1980)**.
12. P. Y. Lee, Lanzhou Lectures on Hestock Integration, World Scientific, Singapore **(1989)**.
13. Manuharawati, Kekontinuan mutlak suatu fungsi bernilai real dengan pendekatan selang-δDasar, *Prosiding Seminar Nasional Matematika dan Pendidikan Matematika*, Unesa, Surabaya **(2013)**.
14. Manuharawati, *International Mathematical Forum* 11, 591 **(2016)**, http://dx.doi.org/10.12988/imf.2016.644.
15. S. Schwabik, Generalized Ordinary Differential Equations, World Scientific, Singapore **(1992)**.
16. Tohoku, *Math. J.* 1, 95 **(1949)**.

Received: 20 August 2016. Accepted: 22 May 2017.